

# Bertrand Spaces and Projected Closed Orbits in General Relativity

M. Rahimkhanli<sup>1</sup>

and

N. Riazi<sup>2</sup>

<sup>1</sup>*Physics Department and Biruni Observatory,*

*Shiraz University, Shiraz 71454, Iran*

<sup>2</sup>*Department of Physics, Shahid Beheshti University,*

*Evin, Tehran 19839, Iran*

In the present work, metrics which lead to projected closed orbits are found by comparing the relativistic differential equation of orbits with the corresponding classical differential equation. Physical and geometrical properties of these peculiar spacetimes are derived and discussed. It is also shown that some of these spacetimes belong to the broader class of the Bertrand spacetimes.

**Keywords:** orbit theory, Bertrand's theorem, Bertrand spacetimes

## I. INTRODUCTION

In classical mechanics, the existence of closed orbits is very important, since it guides us toward discovering further symmetries of the dynamical system. In general, closed-ness of orbits for a specific angular momentum depends on energy. In order to check the long term behavior, the stability of orbits should also be considered [1–3]. Classical Bertrand's theorem states that only two types of potentials produce stable, closed orbits: the Kepler potential and the harmonic oscillator potential [1, 4].

Classical laws of gravitation in Newtonian mechanics, gained many successes in describing the gravitational phenomena, but they couldn't explain some of the observations such as the precession of the planetary orbits. Hence, a successful understanding of the gravitation required a new approach. This approach was introduced in the form of the general theory of relativity by Albert Einstein in 1915. Precession of the orbits is explained by this theory, that of Mercury around the Sun being the most famous example. One of the most important solutions of the Einstein's field equations, is the Schwarzschild solution. Within this metric, orbits are not closed. This example shows that the general relativistic bound orbits in vacuum are not generally closed, but rather they form precessing ellipses, as projected onto a spacelike hyper surface [5–7].

In 1992, Perlick [8] showed that Bertrand's theorem can be reformulated in general relativity. This proposition is as the general relativistic analogue of the classical Bertrand theorem and the resulting spacetimes are known as Bertrand spacetimes. Bertrand spacetimes are interesting in their own right at least because of their mathematical properties. From the point of view of manifold theory, closed geodesics have long played a preponderant role in Riemannian geometry. A somewhat similar question, that of characterizing all Riemannian manifolds whose geodesics are all closed, is still wide open [9, 10]. Under certain physical assumptions, the dark matter distribution of some low surface brightness galaxies can be described in terms of a particular class of the Bertrand spacetimes [11].

In this work, we first review the problem of closed orbits in classical mechanics. The equation of orbit in general relativity is presented in section 3. Projected closed orbits in GR are then found by comparing the relativistic equation of orbit with the corresponding classical equation (section 4). In section 5, the physical and geometrical properties of the resulting spacetimes are derived and discussed. We introduce the Bertrand spacetimes and discuss their relevance to some of the spacetimes found in the present work in section 6. The final section contains our concluding remarks.

## II. EQUATIONS OF ORBIT IN CLASSICAL MECHANICS

Closed orbits appear in central forces and potentials in dynamics. We restrict ourselves to conservative central forces, where the potential  $V(r)$  is a function of  $r$  only. Since the potential energy (and thus the Hamiltonian) involves only the radial distance, the problem has spherical symmetry and the total angular momentum vector  $\mathbf{L}$ , is conserved:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad (1)$$

$$|\mathbf{L}| = mr^2\dot{\varphi} \equiv L = \text{constant}, \quad (2)$$

where  $\mathbf{r}$  is the radial vector,  $\mathbf{p}$  is the linear momentum,  $m$  is the test mass,  $\varphi$  is angular coordinate in the plane of orbit, and  $L$  is the magnitude of angular momentum. Since the force is conservative, on the basis of the general energy conservation theorem, the total energy  $E$ , is a constant of motion:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = \text{constant}. \quad (3)$$

Consider a test particle of unit mass ( $m = 1$ ) moving in a conservative, spherically symmetric, attractive potential within the framework of classical mechanics. The total energy and angular

momentum of this particle are constants of motion. The resulting orbit is then confined to a plane, which we conveniently identify with the (x,y)-plane (i.e.  $\theta = \frac{\pi}{2}$ ). After a little algebra, we arrive at the following equation of orbit:

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{2}{L^2} V - \frac{2E}{L^2} = 0, \quad (4)$$

in which  $L$  and  $E$  are constants and  $u = \frac{1}{r}$ . For the Kepler potential,  $V = -\frac{1}{r} = -u$ , (we assume the potential coefficients  $k = 1$ ), the equation of orbit 4 becomes

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 - \frac{2}{L^2} u - \frac{2E}{L^2} = 0. \quad (5)$$

By taking the derivative of equation 5, a simpler, linear equation results

$$\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2}, \quad (6)$$

which has the following closed orbit solution [1]:

$$u = \frac{1}{r} = \frac{1}{L^2} [1 + \sqrt{1 + 2EL^2} \cos(\varphi - \varphi_0)], \quad (7)$$

in which  $\varphi_0$  is the initial value of  $\varphi$ . The orbits are ellipses with one focus located at the center of potential  $r = 0$  (Kepler's first law). For the classical motion of a unit mass test particle inside a 3D harmonic potential,  $V = \frac{1}{2}r^2 = \frac{1}{2u^2}$ , (we assume the potential coefficients  $k' = 1$ ), the equation of orbit 4 becomes

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{1}{L^2} \frac{1}{u^2} - \frac{2E}{L^2} = 0. \quad (8)$$

Equation 8, after taking a derivative becomes

$$\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2} \frac{1}{u^3}, \quad (9)$$

which has the following closed orbit solution:

$$u^2 = \frac{1}{r^2} = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi, \quad (10)$$

in which  $a^{-1}$  and  $b^{-1}$  are semi-major and semi-minor axes of the ellipse, respectively. Here, like in the Kepler problem, the orbits are also ellipses, but this time with center at  $r = 0$ . According to the Bertrand's theorem, the Kepler and harmonic potentials are the only attractive potentials in classical mechanics which lead to closed bound orbits [1, 4]. These potentials and only these, could possibly produce closed orbits for any arbitrary combination of  $L$  and  $E$  ( $E < 0$ ) [1, 4, 12]. The

proof of Bertrand's theorem is not difficult, and has actually been included (with various levels of rigor) in several textbooks and papers (for example see [4] or [13]).

The closed-ness of orbits in classical mechanics signals the existence of extra constants of motion besides the total energy and angular momentum. It can be shown that for the Kepler problem, the following vector (known as the Laplace-Runge-Lenz vector) is a constant of motion [1]:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}}, \quad (11)$$

where  $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$ . This vector has become known amongst physicists as the Runge-Lenz vector, but priority belongs to Laplace [1]. The Laplace-Runge-Lenz vector for the 3D harmonic potential reads [14]

$$\mathbf{A} = f(\mathbf{r}, L, E, \omega) [\mathbf{p} \times \mathbf{L}] - g(\mathbf{r}, L, E, \omega) \hat{\mathbf{r}}, \quad (12)$$

in which  $\omega$  is the frequency of the oscillator;  $f$  and  $g$  are certain functions of  $\mathbf{r}$  and constants of motion.

### III. EQUATIONS OF ORBIT IN GENERAL RELATIVITY

Let us now turn to general relativity. In GR, equations of motion of a freely falling particle, known as geodesic equations, are [5–7]:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\kappa} \frac{dx^\nu}{d\lambda} \frac{dx^\kappa}{d\lambda} = 0, \quad (13)$$

where  $\Gamma^\mu_{\nu\kappa}$  is the affine connection and  $\lambda$  is the affine parameter. Consider orbits in a static, spherically symmetric spacetime [5]:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \quad (14)$$

By using this metric, the geodesic equations 13 take the following forms [5]:

$$\frac{d^2 r}{d\lambda^2} + \frac{A'}{2A} \left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{A} \left(\frac{d\theta}{d\lambda}\right)^2 - \frac{r \sin^2 \theta}{A} \left(\frac{d\varphi}{d\lambda}\right)^2 + \frac{B'}{2A} \left(\frac{dt}{d\lambda}\right)^2 = 0, \quad (15)$$

$$\frac{d^2 \theta}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} - \sin \theta \cos \theta \left(\frac{d\varphi}{d\lambda}\right)^2 = 0, \quad (16)$$

$$\frac{d^2 \varphi}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\varphi}{d\lambda} + 2 \cot \theta \frac{d\theta}{d\lambda} \frac{d\varphi}{d\lambda} = 0, \quad (17)$$

$$\frac{d^2 t}{d\lambda^2} + \frac{B'}{B} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0, \quad (18)$$

where prime denotes  $d/dr$ . Since the field is isotropic, we may consider the orbit of our particle to be confined to the equatorial plane, that is  $\theta = \frac{\pi}{2}(= \text{const.})$  [5, 6]. Thus, the geodesic equations become:

$$\frac{d^2 r}{d\lambda^2} + \frac{A'}{2A} \left( \frac{dr}{d\lambda} \right)^2 - \frac{r}{A} \left( \frac{d\varphi}{d\lambda} \right)^2 + \frac{B'}{2A} \left( \frac{dt}{d\lambda} \right)^2 = 0, \quad (19)$$

$$\frac{d^2 \varphi}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\varphi}{d\lambda} = 0, \quad (20)$$

$$\frac{d^2 t}{d\lambda^2} + \frac{B'}{B} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0. \quad (21)$$

Now, we are going to obtain the relativistic equations of the orbit. We can rewrite relations 20 and 21 in the following forms:

$$\frac{d^2 \varphi}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\varphi}{d\lambda} = \frac{1}{r^2} \frac{d}{d\lambda} \left( r^2 \frac{d\varphi}{d\lambda} \right) = 0, \quad (22)$$

$$\frac{d^2 t}{d\lambda^2} + \frac{B'}{B} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = \frac{1}{B} \frac{d}{d\lambda} \left( B \frac{dt}{d\lambda} \right) = 0. \quad (23)$$

From above, there result two constants of motion

$$r^2 \frac{d\varphi}{d\lambda} \equiv J = \text{constant}, \quad (24)$$

$$B \frac{dt}{d\lambda} = \text{constant}. \quad (25)$$

Since  $\lambda$  is an arbitrary affine parameter, we can set the constant term  $B \frac{dt}{d\lambda}$  equal to 1:

$$B \frac{dt}{d\lambda} = 1. \quad (26)$$

From 24 we see that  $J$  (that is angular momentum per unit mass of particle) is a constant of motion. Using relations 24 and 26 in the geodesic equation 19, and also the following identities:

$$\frac{d}{d\lambda} \left[ A \left( \frac{dr}{d\lambda} \right)^2 \right] = 2A \frac{dr}{d\lambda} \frac{d^2 r}{d\lambda^2} + A' \left( \frac{dr}{d\lambda} \right)^3, \quad (27)$$

$$\frac{d}{d\lambda} \left[ \frac{J^2}{r^2} - \frac{1}{B} \right] = -\frac{2J^2}{r^3} \frac{dr}{d\lambda} + \frac{B'}{B^2} \frac{dr}{d\lambda}, \quad (28)$$

we arrive at

$$\frac{d}{d\lambda} \left[ A \left( \frac{dr}{d\lambda} \right)^2 + \frac{J^2}{r^2} - \frac{1}{B} \right] = 0. \quad (29)$$

Equation 29, shows that the quantity  $A\left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B}$  is another constant of motion:

$$A\left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} \equiv -\varepsilon. \quad (30)$$

From equation 30, one easily obtains the following equation for the orbit in general relativity:

$$\left(\frac{du}{d\varphi}\right)^2 + \frac{1}{A}u^2 - \frac{1}{J^2AB} + \frac{\varepsilon}{J^2A} = 0, \quad (31)$$

in which  $J$  and  $\varepsilon$  are constants,  $u = \frac{1}{r}$ , and  $A$  and  $B$  are functions of  $r$  only. The Schwarzschild solution in the Boyer-Lindquist form ( $G = c = 1$ ) is given by [6]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2. \quad (32)$$

For this metric (with unit mass  $M = 1$ ), the functions  $A(r)$  and  $B(r)$  read:

$$A(r) = (1 - 2u)^{-1}, \quad B(r) = (1 - 2u) = A^{-1}, \quad (33)$$

and the equation of orbit 31, takes the following form:

$$\left(\frac{du}{d\varphi}\right)^2 - 2u^3 + u^2 - \frac{2\varepsilon}{J^2}u + \frac{1}{J^2}(1 + \varepsilon) = 0. \quad (34)$$

It can be shown that this equation has the following approximate solution [6]:

$$u \simeq \frac{\varepsilon}{J^2}[1 + e \cos((1 - \delta)\varphi)], \quad (35)$$

where  $e$  is eccentricity of the orbit, and  $\delta = \frac{2\varepsilon}{J^2}$  is constant. This result shows the famous precession of the perihelion, which implies that general relativistic bound orbits in vacuum are not generally closed, but rather they form precessing ellipses, that of Mercury around the Sun being the most famous example [5–7].

#### IV. PROJECTED CLOSED ORBITS IN GR

Now we have both classical 4 and relativistic 31 differential equations of orbit at hand. We know that the classical equation, with the Kepler potential and the harmonic potential, lead to closed orbits. We are going to find metrics which lead to closed orbits within GR, by comparing the relativistic differential equation of orbits with the corresponding classical differential equation. Obviously, we have to go beyond the Schwarzschild vacuum solution for this purpose. We can compare the GR 31 and Newtonian 4 equations in several different ways. One way is that we set the second term of the equation 31 equal to the second term of the equation 4:

$$\frac{1}{A}u^2 = u^2 \quad (36)$$

or equivalently:

$$A(r) = 1. \quad (37)$$

Now we obtain  $B(r)$ , by setting  $A = 1$  in the relativistic differential equation 31, and then setting the resulting equation equal to classical differential equation 4. This leads to

$$B(r) = \frac{\left(-\frac{L^2}{2J^2}\right)}{V(r) - \left[E + \frac{L^2}{2J^2}\varepsilon\right]}. \quad (38)$$

In other words, we demand the GR orbit equation 31 to take apparently the form of the classical orbit equation 4. Now, if we insert the Kepler potential, for  $V(r)$  in 38, then we turn the relativistic differential equation of orbits to the differential equation of orbits which have exactly the same form as the classical Kepler orbits (i.e. ellipses with one focus located at the center). By setting  $V(r) = -\frac{1}{r}$  in 38,  $B(r)$  takes the following form:

$$B(r) = C \frac{r}{r + C_1}, \quad (39)$$

in which  $C = \left(\frac{2J^2}{L^2}E + \varepsilon\right)^{-1}$  and  $C_1 = \left(E + \frac{L^2}{2J^2}\varepsilon\right)^{-1}$  are constants. From 37 and 39, the metric 14 reads

$$ds^2 = -\frac{r}{r + C_1}dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \quad (40)$$

where the constant  $C$  is absorbed into the definition of  $t$ . Since the metric 40 leads to exactly the same equation of the orbit as the classical Kepler orbits, we have the elliptical orbits. Similarly, if we set the harmonic potential, for  $V(r)$  in 38, then we arrive from the relativistic differential equation of orbits to one which is exactly the same differential equation of classical harmonic orbits (i.e. ellipses centered at  $r=0$ ). By setting  $V(r) = \frac{1}{2}r^2$  in 38,  $B(r)$  takes the following form:

$$B(r) = C \frac{1}{1 + C_2r^2}, \quad (41)$$

in which  $C_2 = -(2E + \frac{L^2}{J^2}\varepsilon)^{-1}$  is constant. From 37 and 41, the metric 14 reads

$$ds^2 = -\frac{1}{1 + C_2r^2}dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \quad (42)$$

where again the constant  $C$  is absorbed into the definition of  $t$ . Since the metric 42 leads to exactly the same equation of orbit as the classical harmonic orbits, we have the elliptical orbits in this spacetime too.

We pointed out that we can compare the relativistic differential equation of orbit 31, with its

TABLE I: All possible spacetimes which obtained by using form-compatibility method.

<i>metric</i>	$B(r)$	$A(r)$	<i>metric</i>	$B(r)$	$A(r)$
1	$\frac{r}{r+C_1}$	1	10	$\frac{r^2+C_{13}}{r^4+C_{11}r^2}$	$\frac{r^2+C_{11}}{r^2+C_{13}}$
2	$\frac{1}{1+C_2r^2}$	1	11	$\frac{r+C_{14}}{r^2+C_{11}}$	$\frac{r^2+C_{11}}{r+C_{14}}$
3	$\frac{r^3}{r^3+C_3r^2+C_4}$	$\frac{1}{r}$	12	$\frac{r^4+C_{15}}{r^2+C_{11}}$	$\frac{r^2+C_{11}}{r^4+C_{15}}$
4	$\frac{r^6}{r^6+C_5r^2+C_6}$	$\frac{1}{r^4}$	13	$\frac{r^2}{r^4+C_{10}r^3+C_8}$	$r^2$
5	$\frac{r^4}{r^4+C_7r+C_8}$	$\frac{1}{r^2}$	14	$\frac{r^2}{r^6+C_{12}r^4+C_6}$	$r^2$
6	$\frac{r^4}{r^4+C_9}$	$\frac{1}{r^2}$	15	$\frac{r^2}{r^3+C_{13}r+C_7}$	$r$
7	$\frac{r^4+C_{10}r^3}{r^2+C_{11}}$	$\frac{r^2+C_{11}}{r^2+C_{10}r}$	16	$\frac{r^4}{r^2+C_{16}}$	$\frac{1}{r^2}$
8	$\frac{r^6+C_{12}r^4}{r^2+C_{11}}$	$\frac{r^2+C_{11}}{r^4+C_{12}r^2}$	17	$\frac{r^2}{r+C_{17}}$	1
9	$\frac{r^3+C_{13}r}{r^2+C_{11}}$	$\frac{r^2+C_{11}}{r^2+C_{13}}$	18	$\frac{r^2}{r^4+C_{18}}$	1

TABLE II: The constants which are used in the definition of the metrics.

$C_1$	$(E + \frac{L^2}{2J^2}\epsilon)^{-1}$	$C_7$	$\frac{-J^2}{E\epsilon}$	$C_{13}$	$\frac{-L^2}{2E}$
$C_2$	$-(2E + \frac{L^2}{J^2}\epsilon)^{-1}$	$C_8$	$\frac{L^2J^2}{2E\epsilon}$	$C_{14}$	$\frac{-L^2}{2}$
$C_3$	$\frac{-EJ^2}{\epsilon}$	$C_9$	$(\frac{2E\epsilon}{L^2J^2} + \frac{1}{L^2})^{-1}$	$C_{15}$	$L^2$
$C_4$	$\frac{L^2J^2}{2\epsilon}$	$C_{10}$	$\frac{1}{E}$	$C_{16}$	$\frac{-L^2\epsilon}{2E\epsilon+J^2}$
$C_5$	$\frac{2EJ^2}{\epsilon}$	$C_{11}$	$\frac{J^2}{\epsilon}$	$C_{17}$	$\frac{-J^2E}{\epsilon} - \frac{L^2}{2}$
$C_6$	$\frac{-L^2J^2}{\epsilon}$	$C_{12}$	$-2E$	$C_{18}$	$\frac{2J^2E}{\epsilon} + L^2$

classical counterpart 4, in several different ways. Until now, we derived two of the possible options. As one can see from equations 31 and 4, there are three terms in the relativistic differential equation which can be compared with the three terms in the classical equation, i.e. we can arrange the correspondence in 9 different ways. Since for each choice there are two possible potentials (Kepler and harmonic), we have a total 18 possible spacetimes using our form-compatibility method. Hitherto, we set the second term of the equation 31 equal to the second term of the equation 4, and the functions  $A$  and  $B$  read as 37 and 38, then by using the Kepler and harmonic potentials, metric 40 and metric 42 were derived. Similary, we can set the second term of the equation 31 equal to the third or fourth term of the equation 4, and so on. All 18 possible spacetimes derived in this way, are listed in table I.



## V. PHYSICAL AND GEOMETRICAL STRUCTURE OF THE SPACETIMES

In this section, we calculate the geometrical tensors of the spacetimes obtained in the previous section and discuss some of their properties.

The metric 40, is asymptotically flat (Minkowski). For this metric the components of the Einstein tensor read

$$(G^\mu{}_\nu) = \text{diag}\left(0, \frac{C_1}{r^2(r+C_1)}, \frac{C_1(C_1-2r)}{4r^2(r+C_1)^2}, \frac{C_1(C_1-2r)}{4r^2(r+C_1)^2}\right), \quad (43)$$

and the Ricci scalar and the Kretschmann invariant read

$$R = -\frac{3C_1^2}{2r^2(r+C_1)^2}, \quad K = \frac{3C_1^2(8r^2+8C_1r+3C_1^2)}{4r^4(r+C_1)^4}. \quad (44)$$

Note that the Einstein tensor, the Ricci scalar and the Kretschmann invariant are singular at  $r=0$  and  $r=-C_1$ , and vanish as  $r \rightarrow \infty$ . From the form of the Einstein tensor through the Einstein equations ( $G^\mu{}_\nu = -8\pi T^\mu{}_\nu$ ), we conclude that the following density and pressure components are required to support the metric:

$$\rho = 0, p_r = -\frac{1}{8\pi}G^1{}_1 = -\frac{C_1}{8\pi r^2(r+C_1)}, p_t = -\frac{1}{8\pi}G^2{}_2 = -\frac{C_1(C_1-2r)}{32\pi r^2(r+C_1)^2}. \quad (45)$$

It is seen that the energy density vanishes, and the weak energy condition [15, 16] is violated throughout the spacetime:

$$\rho + p_r < 0 \quad (46)$$

(for  $C_1 > 0$ )

$$\rho + p_t < 0 \quad (47)$$

(for  $C_1 < 0$ ).

The line element 40, (if  $C_1 < 0$ ), has a singularity [6, 17, 18] at  $r_m = -C_1$ . This singularity is a curvature singularity, since the Kretschmann invariant 44, is infinite at  $r=-C_1$ . For  $r < r_m$ , the metric signature becomes improper (i.e. it becomes non-Lorentzian or Euclidean).

Tolman mass-energy of a physical system, is given by the Tolman formula [19]

$$M = \int_V (-T^0{}_0 + T^1{}_1 + T^2{}_2 + T^3{}_3) \sqrt{-g} dV, \quad (48)$$

in which  $T^i_j$  is the energy-momentum-stress tensor of the system, and  $g$  is determinant of the metric tensor. Thus, total mass of a spherically symmetric static spacetime is given by

$$M = \int_0^{2\pi} \int_0^\pi \int_0^\infty (-T^0_0 + T^1_1 + T^2_2 + T^3_3) \sqrt{-g} dr d\theta d\varphi, \quad (49)$$

where for the spherically symmetric static spacetime 14,  $\sqrt{-g}$  reads

$$\sqrt{-g} = \sqrt{A(r)B(r)} r^2 \sin\theta. \quad (50)$$

From 49, 45, and 40, total mass of the spacetime reads

$$M = -\frac{1}{2}C_1 \quad (51)$$

Note that for  $C_1 < 0$ , the total mass is positive.

Metric 42, is asymptotically non-flat. For this metric, the components of the Einstein tensor, the Ricci scalar and the Kretschmann invariant read

$$(G^\mu_\nu) = \text{diag}(0, -\frac{2C_2}{1+C_2r^2}, -\frac{C_2(2-C_2r^2)}{(1+C_2r^2)^2}, -\frac{C_2(2-C_2r^2)}{(1+C_2r^2)^2}), \quad (52)$$

$$R = \frac{6C_2}{(1+C_2r^2)^2}, \quad K = \frac{12C_2^2(1+2C_2^2r^4)}{(1+C_2r^2)^4}. \quad (53)$$

These geometrical quantities are regular everywhere if  $C_2 > 0$ , and asymptotically tend to constants as  $r \rightarrow 0$  or  $r \rightarrow \infty$ . The energy density and pressure components for this metric read

$$\rho = 0, p_r = -\frac{1}{8\pi}G^1_1 = \frac{C_2}{4\pi(1+C_2r^2)}, p_t = -\frac{1}{8\pi}G^2_2 = \frac{C_2(2-C_2r^2)}{8\pi(1+C_2r^2)^2}. \quad (54)$$

We see that for  $r > \sqrt{\frac{2}{C_2}}$ , WEC is violated throughout the spacetime:

$$\rho + p_t < 0, \quad (55)$$

while for  $r < \sqrt{\frac{2}{C_2}}$ , although the energy density vanishes, but the pressure components satisfy the weak energy condition:

$$\rho = 0, \quad \rho + p_i \geq 0. \quad (56)$$

From 54, for  $C_2 \geq 0$  we have

$$\rho + p_r + p_t + p_t = \frac{3C_2}{4\pi(1+C_2r^2)^2} \geq 0. \quad (57)$$

From 57 and 56, we see that the strong energy condition [16, 18] is satisfied for  $r < \sqrt{\frac{2}{C_2}}$  if  $C_2 > 0$ . Thus, for this spacetime ( $C_2 > 0$ ) there is a radius ( $r_h = \sqrt{\frac{2}{C_2}}$ ) below which the WEC and SEC

are satisfied.

If  $C_2 < 0$ , the line element 42 has a singularity at  $r_m = \frac{1}{\sqrt{-C_2}}$ . Since the Kretschmann invariant 53, is infinite at  $r = \frac{1}{\sqrt{-C_2}}$ , this singularity is a curvature singularity.

From 49, 54, and 42, total mass of the spacetime reads

$$M = \frac{1}{\sqrt{C_2}}. \quad (58)$$

We see that for  $C_2 > 0$ , the total mass is real and positive. Therefore this metric may be a perfect fluid solution of the Einstein's field equations.

For the metric 3 to the metric 18 in table I, components of the Einstein tensor and the Ricci scalar are given in table III and table IV respectively. Some properties of these spacetimes, are given in table V.

TABLE III: The components of the Einstein tensor for all the 18 classes of spacetimes.

metric	$\frac{-1}{8\pi}(G^\mu{}_\nu) [= diag(\rho, p_r, p_t, p_t)]$
1	$diag(0, \frac{-C_1}{8\pi r^2(r+C_1)}, \frac{C_1(2r-C_1)}{32\pi r^2(r+C_1)^2}, \frac{C_1(2r-C_1)}{32\pi r^2(r+C_1)^2})$
2	$diag(0, \frac{2C_2}{8\pi(1+C_2r^2)}, \frac{C_2(2-C_2r^2)}{8\pi(1+C_2r^2)^2}, \frac{C_2(2-C_2r^2)}{8\pi(1+C_2r^2)^2})$
3	$diag(\frac{1-2r}{8\pi r^2}, \frac{-2C_3r^3+C_3r^2-r^4+r^3-4C_4r+C_4}{8\pi r^2(r^3+C_3r^2+C_4)},$ $\frac{-3C_3r^5-4C_3^2r^4-6C_3C_4r^2+11C_4r^3-14C_4^2-2r^6}{32\pi r(r^3+C_3r^2+C_4)^2},$ $\frac{-3C_3r^5-4C_3^2r^4-6C_3C_4r^2+11C_4r^3-14C_4^2-2r^6}{32\pi r(r^3+C_3r^2+C_4)^2})$
4	$diag(\frac{1-5r^4}{8\pi r^2}, \frac{-5C_5r^6+C_5r^2-r^{10}+r^6-7C_6r^4+C_6}{8\pi r^2(r^6+C_5r^2+C_6)},$ $\frac{r^2(-10C_5^2r^4-24C_5C_6r^2+8C_6r^6-17C_6^2-2r^{12})}{8\pi(r^6+C_5r^2+C_6)^2},$ $\frac{r^2(-10C_5^2r^4-24C_5C_6r^2+8C_6r^6-17C_6^2-2r^{12})}{8\pi(r^6+C_5r^2+C_6)^2})$
5	$diag(\frac{1-3r^2}{8\pi r^2}, \frac{-4C_7r^3+C_7r-r^6+r^4-5C_8r^2+C_8}{8\pi r^2(r^4+C_7r+C_8)},$ $\frac{4C_7r^5-19C_7^2r^2-44C_7C_8r+16C_8r^4-28C_8^2-4r^8}{32\pi(r^4+C_7r+C_8)^2},$ $\frac{4C_7r^5-19C_7^2r^2-44C_7C_8r+16C_8r^4-28C_8^2-4r^8}{32\pi(r^4+C_7r+C_8)^2})$
6	$diag(\frac{1-3r^2}{8\pi r^2}, \frac{-5C_9r^2+C_9-r^6+r^4}{8\pi r^2(r^4+C_9)}, \frac{4C_9r^4-7C_9^2-r^8}{8\pi(r^4+C_9)^2}, \frac{4C_9r^4-7C_9^2-r^8}{8\pi(r^4+C_9)^2})$
7	$diag(\frac{C_{11}(-2C_{10}r+C_{11}-r^2)}{8\pi r^2(r^2+C_{11})^2}, \frac{C_{11}^2-4C_{10}C_{11}r-3C_{11}r^2-2C_{10}r^3-2r^4}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-2r^5-8C_{11}r^3-14C_{11}^2r+C_{10}r^4+2C_{10}C_{11}r^2-7C_{10}C_{11}^2}{16\pi r(r^2+C_{11})^3},$ $\frac{-2r^5-8C_{11}r^3-14C_{11}^2r+C_{10}r^4+2C_{10}C_{11}r^2-7C_{10}C_{11}^2}{16\pi r(r^2+C_{11})^3})$
8	$diag(\frac{-3C_{11}C_{12}r^2-C_{12}r^4+C_{11}^2-5C_{11}r^4+r^4+2C_{11}r^2-3r^6}{8\pi r^2(r^2+C_{11})^2},$ $\frac{C_{11}^2-5C_{11}C_{12}r^2+2C_{11}r^2-7C_{11}r^4-3C_{12}r^4+r^4-5r^6}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-7r^6-20C_{11}r^4-17C_{11}^2r^2-C_{12}r^4-4C_{11}C_{12}r^2-7C_{11}^2C_{12}}{8\pi(r^2+C_{11})^3},$ $\frac{-7r^6-20C_{11}r^4-17C_{11}^2r^2-C_{12}r^4-4C_{11}C_{12}r^2-7C_{11}^2C_{12}}{8\pi(r^2+C_{11})^3})$
9	$diag(\frac{-C_{11}C_{13}+C_{13}r^2+C_{11}^2-C_{11}r^2}{8\pi r^2(r^2+C_{11})^2}, \frac{C_{11}^2-2C_{11}r^2-2C_{11}C_{13}-r^4}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-r^6-4C_{11}r^4-19C_{11}^2r^2+C_{13}r^4+16C_{11}C_{13}r^2-C_{11}^2C_{13}}{32\pi r^2(r^2+C_{11})^3},$ $\frac{-r^6-4C_{11}r^4-19C_{11}^2r^2+C_{13}r^4+16C_{11}C_{13}r^2-C_{11}^2C_{13}}{32\pi r^2(r^2+C_{11})^3})$

10	$diag(\frac{-C_{11}C_{13}+C_{13}r^2+C_{11}^2-C_{11}r^2}{8\pi r^2(r^2+C_{11})^2}, \frac{C_{11}^2+C_{11}r^2+C_{11}C_{13}+2r^4+3C_{13}r^2}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-r^6+2C_{11}r^4-5C_{13}r^4-2C_{11}C_{13}r^2-C_{11}^2r^2-C_{11}^2C_{13}}{8\pi r^2(r^2+C_{11})^3},$ $\frac{-r^6+2C_{11}r^4-5C_{13}r^4-2C_{11}C_{13}r^2-C_{11}^2r^2-C_{11}^2C_{13}}{8\pi r^2(r^2+C_{11})^3})$
11	$diag(\frac{-2C_{11}r+C_{14}r^2+r^4+2C_{11}r^2+C_{11}^2-C_{11}C_{14}}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-2C_{11}r+C_{14}r^2+r^4+2C_{11}r^2+C_{11}^2-C_{11}C_{14}}{8\pi r^2(r^2+C_{11})^2},$ $\frac{3C_{11}C_{14}r-C_{14}r^3+3C_{11}r^2-C_{11}^2}{8\pi r(r^2+C_{11})^3}, \frac{3C_{11}C_{14}r-C_{14}r^3+3C_{11}r^2-C_{11}^2}{8\pi r(r^2+C_{11})^3})$
12	$diag(\frac{-3r^6-5C_{11}r^4+C_{15}r^2+r^4+2C_{11}r^2+C_{11}^2-C_{11}C_{15}}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-3r^6-5C_{11}r^4+C_{15}r^2+r^4+2C_{11}r^2+C_{11}^2-C_{11}C_{15}}{8\pi r^2(r^2+C_{11})^2},$ $\frac{-C_{15}r^2+3C_{11}C_{15}-9C_{11}r^4-10C_{11}^2r^2-3r^6}{8\pi(r^2+C_{11})^3},$ $\frac{-C_{15}r^2+3C_{11}C_{15}-9C_{11}r^4-10C_{11}^2r^2-3r^6}{8\pi(r^2+C_{11})^3})$
13	$diag(\frac{r^2+1}{8\pi r^4}, \frac{C_{10}r^5+r^6+r^4+C_8r^2-3C_8}{8\pi r^4(r^4+C_{10}r^3+C_8)},$ $\frac{-4r^8+C_{10}^2r^6+32C_8C_{10}r^3+4C_8^2+48C_8r^4}{32\pi r^4(r^4+C_{10}r^3+C_8)^2},$ $\frac{-4r^8+C_{10}^2r^6+32C_8C_{10}r^3+4C_8^2+48C_8r^4}{32\pi r^4(r^4+C_{10}r^3+C_8)^2})$
14	$diag(\frac{r^2+1}{8\pi r^4}, \frac{C_{12}r^6+C_{12}r^4+r^8+3r^6+C_6r^2-3C_6}{8\pi r^4(r^6+C_{12}r^4+C_6)},$ $\frac{-5r^{12}-3C_{12}r^{10}+23C_6r^6-C_{12}^2r^8+C_6^2+12C_6C_{12}r^4}{8\pi r^4(r^6+C_{12}r^4+C_6)^2},$ $\frac{-5r^{12}-3C_{12}r^{10}+23C_6r^6-C_{12}^2r^8+C_6^2+12C_6C_{12}r^4}{8\pi r^4(r^6+C_{12}r^4+C_6)^2})$
15	$diag(\frac{1}{8\pi r^2}, \frac{C_{13}r^2-2C_{13}r+r^4+C_7r-3C_7}{8\pi r^3(r^3+C_{13}r+C_7)},$ $\frac{14C_{13}r^3+2C_{13}^2r+5C_7C_{13}+27C_7r^2}{32\pi r^2(r^3+C_{13}r+C_7)^2}, \frac{14C_{13}r^3+2C_{13}^2r+5C_7C_{13}+27C_7r^2}{32\pi r^2(r^3+C_{13}r+C_7)^2})$
16	$diag(\frac{1-3r^2}{8\pi r^2}, \frac{-5C_{16}r^2+C_{16}-3r^4+r^2}{8\pi r^2(r^2+C_{16})}, \frac{-3r^4-7C_{16}r^2-7C_{16}^2}{8\pi(r^2+C_{16})^2},$ $\frac{-3r^4-7C_{16}r^2-7C_{16}^2}{8\pi(r^2+C_{16})^2})$
17	$diag(0, \frac{-r-2C_{17}}{8\pi r^2(r+C_{17})}, \frac{-r^2-2C_{17}r-4C_{17}^2}{32\pi r^2(r+C_{17})^2}, \frac{-r^2-2C_{17}r-4C_{17}^2}{32\pi r^2(r+C_{17})^2})$
18	$diag(0, \frac{2(r^4-C_{18})}{8\pi r^2(r^4+C_{18})}, \frac{-r^8-C_{18}^2+10C_{18}r^4}{8\pi r^2(r^4+C_{18})^2}, \frac{-r^8-C_{18}^2+10C_{18}r^4}{8\pi r^2(r^4+C_{18})^2})$

## VI. RELEVANCE TO BERTRAND SPACETIMES

In this section, we introduce the Bertrand spacetimes, and discuss their relevance to some of the spacetimes of previous sections.

A spacetime is called a “Bertrand spacetime” if it is a spherically symmetric and static spacetime, and there is a circular trajectory through each point and the following inequality is satisfied

$$0 < \frac{rB'(r)}{B(r)} < 1. \quad (59)$$

TABLE IV: The Ricci scalar for all the 18 classes of spacetimes.

<i>metric</i>	<i>R</i>
1	$[-3C_1^2]/[2r^2(r+C_1)^2]$
2	$[-6C_2]/[(1+C_2r^2)^2]$
3	$[-(8r^7 + (17C_3 - 4)r^6 + 4C_3(3C_3 - 2)r^5 + (7C_4 - 4C_3^2)r^4 + 2C_4(13C_3 - 4)r^3 - 8C_3C_4r^2 + 26C_4^2r - 4C_4^2)]/[2r^2(r^3 + C_3r^2 + C_4)^2]$
4	$[-2(5r^{16} + (8C_5 - 1)r^{12} + C_6r^{10} + C_5(15C_5 - 2)r^8 + C_6(35C_5 - 2)r^6 + (23C_6^2 - C_5^2)r^4 - 2C_5C_6r^2 - C_6^2)]/[r^2(r^6 + C_5r^2 + C_6)^2]$
5	$[-(12r^{10} - 4r^8 + 18C_7r^7 + 8C_8r^6 - 8C_7r^5 + (33C_7^2 - 8C_8)r^4 + 74C_7C_8r^3 + 4(11C_8^2 - C_7^2)r^2 - 8C_7C_8r - 4C_8^2)]/[2r^2(r^4 + C_7r + C_8)^2]$
6	$[-2(3r^{10} - r^8 + 2C_9r^6 - 2C_9r^4 + 11C_9^2r^2 - C_9^2)]/[r^2(r^4 + C_9)^2]$
7	$[-4r^6 + C_{10}r^5 + 14C_{11}r^4 + 6C_{10}C_{11}r^3 + 16C_{11}^2r^2 + 13C_{10}C_{11}^2r - 2C_{11}^2]/[r^2(r^2 + C_{11})^3]$
8	$[2(-11r^8 + (1 - 30C_{11} - 3C_{12})r^6 + C_{11}(3 - 23C_{11} - 10C_{12})r^4 + C_{11}^2(3 - 11C_{12})r^2 + C_{11}^3)]/[r^2(r^2 + C_{11})^3]$
9	$[-3r^6 + 3(C_{13} - 4C_{11})r^4 + 3C_{11}(4C_{13} - 7C_{11})r^2 + C_{11}^2(4C_{11} - 7C_{13})]/[2r^2(r^2 + C_{11})^3]$
10	$[2(C_{11} - C_{13})(3r^4 + C_{11}^2)]/[r^2(r^2 + C_{11})^3]$
11	$[2(r^6 + 3C_{11}r^4 + C_{11}r^3 + 3C_{11}(C_{11} + C_{14})r^2 - 3C_{11}^2r + C_{11}^2(C_{11} - C_{14}))]/[r^2(r^2 + C_{11})^3]$
12	$[2(-6r^8 + (1 - 17C_{11})r^6 + 3C_{11}(1 - 5C_{11})r^4 + 3C_{11}(C_{11} + C_{15})r^2 + C_{11}^2(C_{11} - C_{15}))]/[r^2(r^2 + C_{11})^3]$
13	$[4r^8 + 8C_{10}r^7 + 4C_{10}^2r^6 + 6C_{10}r^5 + (3C_{10}^2 + 8C_8)r^4 + 8C_8C_{10}r^3 + 48C_8r^2 + 30C_8C_{10}r + 4C_8^2]/[2r^2(r^4 + C_{10}r^3 + C_8)^2]$
14	$[2(r^{12} + (2C_{12} - 3)r^{10} + C_{12}^2r^8 + 2C_6r^6 + 2C_6(C_{12} + 12)r^4 + 12C_6C_{12}r^2 + C_6^2)]/[r^2(r^6 + C_{12}r^4 + C_6)^2]$
15	$[4r^7 + 8C_{13}r^5 + 2(5C_{13} + 4C_7)r^4 + (4C_{13}^2 + 21C_7)r^3 + 2C_{13}(4C_7 - C_{13})r^2 + C_7(4C_7 - 5C_{13})r - 6C_7^2]/[2r^3(r^3 + C_{13}r + C_7)^2]$
16	$[-2(6r^6 + (14C_{16} - 1)r^4 + C_{16}(11C_{16} - 2)r^2 - C_{16}^2)]/[r^2(r^2 + C_{16})^2]$
17	$[-(3r^2 + 8C_{17}r + 8C_{17}^2)]/[2r^2(r + C_{17})^2]$
18	$[4C_{18}(5r^4 - C_{18})]/[r^2(r^4 + C_{18})^2]$

TABLE V: Some properties of the spacetimes (the sign \* means that the equations are higher than order 3, and no conclusion is reached).

<i>metric</i>	$M$ ( <i>Tolman mass</i> )	$r_m$ ( <i>singularity coord.</i> )	$\lim_{r \rightarrow \infty} R$	<i>Energy Conditions</i> ( <i>WEC &amp; SEC</i> )
1	$-\frac{1}{2}C_1$	$-C_1$ ( <i>if</i> $C_1 < 0$ )	0	<i>violated</i>
2	$\frac{1}{\sqrt{C_2}}$ ( <i>if</i> $C_2 > 0$ )	$\frac{1}{\sqrt{-C_2}}$ ( <i>if</i> $C_2 < 0$ )	0	$r < \sqrt{\frac{2}{C_2}}$ ( $C_2 > 0$ ) <i>WEC &amp; SEC</i> <i>satisfied</i>
3	$\infty$ ( <i>if</i> $C_3 < 0$ ) $-\infty$ ( <i>if</i> $C_3 > 0$ )	$0, \frac{2C_3^2}{3\sqrt[3]{C_{19}}} + \frac{\sqrt[3]{C_{19}}}{6} - \frac{C_3}{3}$ ( $C_{19} = -8C_3^3 + 12\sqrt{81C_4^2 + 12C_4C_3^3} - 108C_4$ )	0	*
4	0	$0, \left( \frac{C_{20}^{\frac{2}{3}} - 12C_5}{(6C_{20})^{\frac{1}{3}}} \right)^{\frac{1}{2}}$ ( $C_{20} = -108C_6 + 12(12C_5^3 + 81C_6^2)^{\frac{1}{2}}$ )	$-\infty$	*
5	0	0, *	<i>const.</i>	*
6	0	0, $\sqrt[4]{-C_9}$ ( <i>if</i> $C_9 < 0$ )	<i>const.</i>	*
7	$-\infty$	0, $-C_{10}$ ( <i>if</i> $C_{10} < 0$ ), $\sqrt{-C_{11}}$ ( <i>if</i> $C_{11} < 0$ )	0	*
8	$-\infty$	0, $\sqrt{-C_{12}}$ ( <i>if</i> $C_{12} < 0$ ), $\sqrt{-C_{11}}$ ( <i>if</i> $C_{11} < 0$ )	<i>const.</i>	*
9	$\infty$	$\sqrt{-C_{11}}$ ( <i>if</i> $C_{11} < 0$ ), $\sqrt{-C_{13}}$ ( <i>if</i> $C_{13} < 0$ )	0	*

Also, it is required that any initial condition for the geodesic equation which is sufficiently close to a circular trajectory gives a periodic trajectory [8].

Bertrand spacetimes are given by the following metrics, which are called types *I* and *II*± [8, 20]

$$ds^2 = -\frac{dt^2}{G + \sqrt{r^{-2} + K}} + \frac{dr^2}{\beta^2(1 + Kr^2)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (\text{Type } I), \quad (60)$$

and

$$ds^2 = -\frac{dt^2}{G \mp r^2[1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4}]^{-1}} + \frac{2(1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4})}{\beta^2((1 - Dr^2)^2 - Kr^4)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (\text{Type } II\pm), \quad (61)$$

10	$\infty$ (if $C_{13} > C_{11}$ ) $-\infty$ (if $C_{13} < C_{11}$ )	$0, \sqrt{-C_{11}}$ (if $C_{11} < 0$ ), $\sqrt{-C_{13}}$ (if $C_{13} < 0$ )	0	*
11	$-\infty$	$\sqrt{-C_{11}}$ (if $C_{11} < 0$ ), $-C_{14}$ (if $C_{14} < 0$ )	0	*
12	$-\infty$	$\sqrt{-C_{11}}$ (if $C_{11} < 0$ ), $\sqrt[4]{-C_{15}}$ (if $C_{15} < 0$ )	<i>const.</i>	*
13	0	*	0	*
14	0	$\left( \frac{C_{21}^{\frac{2}{3}} + 4C_{12}^2 - 2C_{12}C_{21}^{\frac{1}{3}}}{6^{\frac{1}{2}}C_{21}^{\frac{1}{3}}} \right)^{\frac{1}{2}}$ ( $C_{21} = -108C_6 - 8C_{12}^3$ $+12(81C_6^2 + 12C_6C_{12}^3)^{\frac{1}{2}}$ )	0	*
15	$\frac{1}{2}$	$\frac{C_{22}^{\frac{2}{3}} - 12C_{13}}{6C_{22}^{\frac{1}{3}}}$ ( $C_{22} = -108C_7 +$ $12(12C_{13}^3 + 81C_7^2)^{\frac{1}{2}}$ )	0	*
16	$-\infty$	$0, \sqrt{-C_{16}}$ (if $C_{16} < 0$ )	<i>const.</i>	*
17	$-\infty$	$-C_{17}$ (if $C_{17} < 0$ )	0	<i>violated</i>
18	1	$\sqrt[4]{-C_{18}}$ (if $C_{18} < 0$ )	0	$r \geq (5 + 2\sqrt{6})^{\frac{1}{4}} \sqrt[4]{C_{18}}$ $C_{18} \geq 0$ <i>WEC</i> <i>satisfied</i> , $r \geq (\frac{1}{5^{\frac{1}{4}}})^{\frac{1}{4}} \sqrt[4]{C_{18}}$ $C_{18} \geq 0$ <i>SEC</i> <i>satisfied</i> ( <i>WEC</i> & <i>SEC satisfied</i> )

the parameters  $G$ ,  $K$  and  $D$  are real constants, and  $\beta$  is a positive rational number. Conversely, any metric of this form determines a Bertrand spacetime.

According to [20], there are several relevant specific cases of Bertrand spaces: (i) Three classical Riemannian spaces of constant curvature, (ii) Darboux spaces of type III, and (iii) Iwai–Katayama [21] spaces. The classical Riemannian spaces are proven to belong to the Bertrand family  $I$  60, and  $II \pm 61$ , respectively under the identifications [20]

$$\beta = 1, \quad K = -\alpha, \quad (\text{Type } I), \quad (62)$$



$$\beta = 2, \quad K = 0, \quad D = \alpha, \quad (\text{Type } II+), \quad (63)$$

where  $\alpha$  is a constant. And the Darboux spaces and the Iwai–Katayama spaces are proven to belong to the Bertrand family  $II\pm$  61, respectively, under the identifications [20]

$$\beta = 2, \quad K = D^2, \quad D = -\frac{2}{l^2}, \quad (\text{Type } II\pm), \quad (64)$$

$$\beta = \frac{1}{v}, \quad K = D^2, \quad D = -\frac{2b}{a^2}, \quad (\text{Type } II\pm), \quad (65)$$

where  $l$  is an arbitrary constant,  $v$  is a rational number, and  $a$  and  $b$  are two real constants.

It is easy to show that the metric 40 belongs to the Bertrand family  $I$  60 under the identifications

$$\beta = 1, \quad K = 0, \quad (\text{Type } I), \quad (66)$$

( $G = \frac{1}{C_1}$ ). And the metric 42 belongs to the Bertrand family  $II+$  61 under the identifications

$$\beta = 2, \quad K = 0, \quad D = 0, \quad (\text{Type } II+), \quad (67)$$

( $G = \frac{-1}{2C_2}$ ). We therefore see that 40 and 42 are particular cases of the Bertrand spacetimes. We could not identify other metrics of table I with Bertrand spacetimes.

## VII. CONCLUDING REMARKS

We derived metrics for curved spacetimes which lead to closed bound projected orbits by comparing the relativistic differential equation of orbits with the corresponding classical differential equation. We can name this method as the form-compatibility method.

Physical and geometrical properties of these peculiar spacetimes were derived and discussed and it was shown that two of these spacetimes may be perfect fluid solutions of the Einstein's field equations (spacetimes 2 and 18 in table I). It was also shown that two of these spacetimes are particular cases of the Bertrand spacetimes (spacetimes 1 and 2 in table I).

In classical mechanics, the existence of closed orbits guides us toward discovering further symmetries of the dynamical system (as established for the Kepler and harmonic potentials via the existence of the Laplace-Runge-Lenz vector). The Laplace-Runge-Lenz vector is in the direction of the radius vector to the perihelion point on the orbit. Conservation of this vector means that the orientation of the orbit in space is fixed and the orbit stays closed.

In general relativity, the existence of projected closed orbits guides us toward discovering

further symmetries of the spacetime. The geodesic equation of a Bertrand spacetime admits an additional constant of motion related to a non-redundant time-independent second rank Killing tensor field if and only if  $\beta$  is equal either to 1 or to 2 [8]. Two of our spacetimes, have this condition (spacetimes 1 and 2 in table I). All spherically symmetric and static spacetimes which admit non-redundant time independent second rank Killing tensor fields are listed in reference [22]. The corresponding constants of motion are explicitly given in table 2 of this reference. The whole problem of hidden symmetries in general relativity is remains a subject of much interest.

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